Suggested Solutions to Midterm Test for MATH4220

March 10, 2016

1. (**20 points**)

   (a) (**8 points**) Find the general solutions to

   $$u_x - 2u_y + 2u = 0$$

   (b) (**12 points**) Solve the problem:

   $$\begin{cases} y\partial_x u + 3x^2 y\partial_y u = 0 \\ u(x=0,y) = y^2 \end{cases}$$

   Find the region in the xy-plane so that the solution is uniquely determined.

Solution:

(a) **Method 1:Coordinate Method**:
Change variables to

$$x' = x - 2y, \ y' = -2x - y$$

Hence $u_x - 2u_y + 2u = 5u_{x'} + 2u = 0$. Thus the solution is $u(x', y') = f(y')e^{-\frac{2}{5}x'}$, with f an arbitrary function of one variable. Therefore, the general solutions are

$$u(x,y) = f(-2x-y)e^{-\frac{2}{5}(x-2y)}$$

where f is an arbitrary function.

Method 2: Geometric Method
The corresponding characteristic curves are

$$\frac{dx}{1} = \frac{dy}{-2}$$

that is, $y = -2x + C$ where C is an arbitrary constant.Then

$$\frac{d}{dx}u(x, -2x+C) = u_x(x, -2x+C) - 2u(x, -2x+C) = -2u(x, -2x+C)$$

Hence $u(x, -2x+C) = f(C)e^{-2x}$, where f is an arbitrary function. Therefore,

$$u(x,y) = f(2x+y)e^{-2x}$$

where f is an arbitrary function.

(b) The characteristic curves are

$$\frac{dy}{3x^2y} = \frac{dx}{y}$$

that is, $y = x^3 + C$ where C is an arbitrary constant. Then

$$\frac{d}{dx}u(x, x^3+C) = u_x + 3x^2u_y = 0$$

Hence $u(x, x^3 + C) = f(C)$ where f is an arbitrary function. Thus

$$u(x, y) = f(y - x^3)$$

Besides, the auxiliary condition gives that $y^2 = u(x = 0, y) = f(y)$. Hence, the solution is

$$u(x, y) = (y - x^3)^2$$

Note that when $y = 0$ the equation vanishes, thus the characteristic curves break down when $y = 0$, therefore the solution is uniquely determined on $\{(x, y) : y > 0, y > x^3\} \cup \{(x, y) : y < 0, y < x^3\} \cup \{(0, 0)\}$. (Remark: if the solution is **continuous**, then u is uniquely determined on the whole plane by the continuity of u).

2. (**30 points**)

   (a) (**5 points**) State the definition of a well-posed PDE problem.

   (b) (**5 points**) Is the following problem well-posed? Why?

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0, & x^2 + y^2 < 1 \\ \dfrac{\partial u}{\partial \vec{n}}(x, y) = 0, & x^2 + y^2 = 1, \ \vec{n} \text{ is the unit outnorm of } x^2 + y^2 = 1 \end{cases}$$

   (c) (**10 points**) Verifying that $u_n(x, t) = \frac{1}{n} \sin nx e^{-n^2 t}$ solves the following probem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < \pi, \quad -\infty < t < +\infty \\ u(0, t) = u(\pi, t) = 0, & -\infty < t < \infty \\ u(x, t = 0) = \dfrac{1}{n} \sin nx, & 0 \le x \le \pi \end{cases}$$

   for all positive integer n.
   How does the energy change when $t \to \pm\infty$?

   (d) (**10 points**) Is the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < \pi, \quad t < 0 \\ u(0, t) = u(\pi, t) = 0, & t < 0 \\ u(x, t = 0) = 0, & 0 < x < \pi \end{cases}$$

   well-posed ? Why?

Solution:

(a) A PDE problem is said to be well-posed if the following three properties are satisfied:
Existence: There exists at least one solution $u(x, t)$ satisfying all these conditions.
Uniqueness: There is at most one solution.
Stability: The unique solution $u(x, t)$ depends in a stable manner on the data of the problem. This means that if the data are changed a little, the corresponding solution changes only a little.

(b) No.
Let $u(x, t) = C$ where C is an arbitrary constant. Then $\partial_x u = \partial_y u = \partial_x^2 u = \partial_y^2 u = 0$, hence

$$\partial_x^2 u + \partial_y^2 u = 0, x^2 + y^2 < 1$$

$$\frac{\partial u}{\partial \vec{n}} = (\partial_x u, \partial_y u) \cdot (x, y) = 0, x^2 + y^2 = 1$$

Therefore, any constant is the solution of the problem. Hence the solution exists but is not unique.

(c) After a little simple computations, we have for all positive interger n

$$\partial_t u_n(x,t) = -n\sin(nx)e^{-n^2t}$$

$$\partial_x u_n(x,t) = \cos(nx)e^{-n^2t}$$

$$\partial_x^2 u_n(x,t) = -n\sin(nx)e^{-n^2t}$$

then $\partial_t u_n = -n\sin(nx)e^{-n^2t} = \partial_x^2 u_n,\quad 0<x<\pi,\ -\infty<t<\infty$. And

$$u_n(0,t) = 0,\quad -\infty<t<\infty$$

$$u_n(\pi,t) = 0,\quad -\infty<t<\infty$$

$$u_n(x,t=0) = \frac{1}{n}\sin(nx),\quad 0<x<\pi$$

hence u_n is indeed the solution of the problem.
The energy is

$$E = \frac{1}{2}\int_0^\pi |u_n(x,t)|^2 dx = \frac{1}{2n^2}e^{-2n^2t}\int_0^\pi \sin^2(nx)dx = \frac{\pi}{4n^2}e^{-2n^2t}$$

hence $E(t)\to 0$ as $t\to+\infty$ and $E(t)\to+\infty$ as $t\to-\infty$.

(d) No.
On one hand, $u=0$ is a solution of

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0<x<\pi,\quad t<0 \\ u(0,t) = u(\pi,t) = 0, & t<0 \\ u(x,t=0) = 0, & 0<x<\pi \end{cases}$$

On the other hand, $u_n(x,t) = \frac{1}{n}\sin nx e^{-n^2t}$ solves the following probem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0<x<\pi,\quad t<0 \\ u(0,t) = u(\pi,t) = 0, & t<0 \\ u(x,t=0) = \frac{1}{n}\sin nx, & 0\le x\le\pi \end{cases}$$

for all positive integer n by above problem (c).
Note that

$$\int_0^\pi |\frac{1}{n}\sin nx - 0|^2 dx = \frac{\pi}{2n^2} \to 0 \text{ as } n\to+\infty$$

However when $t<0$,

$$\int_0^\pi |\frac{1}{n}\sin(nx)e^{-n^2t} - 0|^2 dx = \frac{\pi}{2n^2}e^{-2n^2t} \to +\infty \text{ as } n\to\infty$$

Hence, when the data $u(x,t=0)$ changes a little in the sense of L^2-norm, the difference of the solutions in L^2-norm tends to infinity. This violates the stability in the sense of L^2-norm, therefore, it is not well-posed.
Remark: consider the stability in uniform sense.
On one hand, $\max_{0<x<\pi}|\frac{1}{n}\sin(nx) - 0| \to 0$ as $n\to\infty$
On the other hand, when $t<0$, $\max_{0<x<\pi}|\frac{1}{n}\sin(nx)e^{-n^2t} - 0| \to +\infty$ as $n\to\infty$
This violates the stability in the uniform sense.

3. (**10 points**) Is there a maximum principle for the Cauchy problem for the 1-dimensional wave equation? Explain why?

Solution: No.

Consider the following Cauchy problem:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0, & -\infty < x < +\infty, \quad t > 0 \\ u(x, t = 0) = 0, & \partial_t u(x, t = 0) = \sin x, \quad -\infty < x < +\infty \end{cases}$$

And the unique solution is given by d'Alembert formula:

$$u(x,t) = \frac{1}{2}\cos(x+t) - \cos(x-t) = -\sin x \sin t, \quad -\infty < x < \infty, t > 0$$

Then $u(x,t)$ attains its maximum 1 only at the interior points $(\frac{\pi}{2} \pm 2n\pi, \frac{3\pi}{2} + 2n\pi)$ or $(\frac{3\pi}{2} \pm 2n\pi, \frac{\pi}{2} + 2n\pi)$ for $n = 0, 1, 2, \cdots$. However, $u(x,t) = 0$ on the boundary $\{(x,t) : t = 0\}$. Therefore there is no maximum principle for the Cauchy problem for the 1-dimensitonal wave equation.

Remark: The key is to find an counterexample.

4. (**10 points**)

   (a) (**5 points**) What is the type of the equation

$$\partial_t^2 u + \partial_{xt}^2 u - 2\partial_x^2 u = 0 \ ?$$

   (b) (**5 points**) Solve the Cauchy problem

$$\begin{cases} \partial_t^2 u - 2\partial_x^2 u = 0, & -\infty < x < +\infty, \quad -\infty < t < +\infty \\ u(x, t = 0) = x^2, & \partial_t u(x, t = 0) = \sin x, \quad -\infty < x < +\infty \end{cases}$$

Solution:

(a) Since $a_{11} = 1, a_{12} = \frac{1}{2}, a_{22} = -2$, then $a_{12}^2 - a_{11}a_{22} = \frac{9}{4} > 0$, hence it is hyperbolic.

(b) The solution is given by d'Alembert Formula directly:

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi y dy$$

Here $c = \sqrt{2}$, $\phi(x) = x^2$ and $\psi(x) \sin x$. Hence

$$u(x,t) = \frac{1}{2}[(x+ct)^2 + (x-ct)^2] + \frac{1}{2\sqrt{2}}\int_{x-ct}^{x+ct} \sin y dy = x^2 + 2t^2 + \frac{1}{\sqrt{2}}\sin x \sin\sqrt{2}t$$

5. (**20 points**) Consider the Cauchy problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & -\infty < x < +\infty, \quad t > 0 \\ u(x, t = 0) = \phi(x) & -\infty < x < +\infty \end{cases}$$

   (a) (**10 points**) Show that any finite energy solution to the Cauchy problem is unique by the energy method.

   (b) (**10 points**) Find the solution with $\phi(x)$ given by

$$\phi(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Solution:

(a) Soppose u_1 and u_2 are two finite energy solution to Cauchy problem. Let $v(x,t) = u_1(x,t) - u_2(x,t)$, then $v(x,t)$ satisfies the following problem:

$$\begin{cases} \partial_t v = \partial_x^2 v, & -\infty < x < +\infty, \quad t > 0 \\ v(x, t=0) = 0 & -\infty < x < +\infty \end{cases}$$

Multiplying the both sides of $\partial_t v = \partial_x^2 v$ by u and taking intergral from $-\infty$ to ∞ with respect to x, then we have

$$\int_{-\infty}^{\infty} \partial_t v v dx = \int_{-\infty}^{\infty} \partial_x^2 v v dx$$

Then

$$L.H.S = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} v^2 dx$$

$$R.H.S = \partial_x v v\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (\partial_x v)^2 dx$$

Note that u_1 and u_2 are finite energy solutions, that is,

$$\frac{1}{2}\int_{-\infty}^{\infty} |u_1(x,t)|^2 dx < +\infty, \quad \frac{1}{2}\int_{-\infty}^{\infty} |u_2(x,t)|^2 dx < +\infty$$

then

$$\frac{1}{2}\int_{-\infty}^{\infty} |v(x,t)|^2 dx \leq \int_{-\infty}^{\infty} |u_1(x,t)|^2 + |u_2(x,t)|^2 dx < +\infty$$

that is, $v(x,t)$ is a finite energy solution which implies that $v(x,t) \to 0$ as $x \to \pm\infty$. Hence

$$R.H.S = -\int_{-\infty}^{\infty} (\partial_x v)^2 dx$$

Then, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} v^2 dx = -\int_{-\infty}^{\infty} (\partial_x v)^2 dx \leq 0$$

and then for $t > 0$

$$0 \leq \int_{-\infty}^{\infty} \frac{1}{2} v^2(x,t) dx \leq \int_{-\infty}^{\infty} \frac{1}{2} v^2(x,0) dx = 0$$

By the continuity of v, we have $v(x,t) \equiv 0$, $-\infty < x < \infty, t > 0$. Thus we have shown that $u_1(x,t) \equiv u_2(x,t)$ for $-\infty < x < \infty$, $t > 0$. Therefore any finite energy solution is unique.

(b) **Method 1:**Find a solution with the from $u(x,t) = U(\frac{x}{\sqrt{4t}})$. Let $p = \frac{x}{\sqrt{4t}}$, then $U(\frac{x}{\sqrt{4t}})$ satisfies the following eqution:

$$U''(p) + 2pU'(p) = 0$$

Hence $U(p) = C_1 + C_2 \int_0^p e^{-s^2} ds$ where C_1, C_2 are arbitrary constants. That is,

$$u(x,t) = U(\frac{x}{\sqrt{4t}}) = C_1 + C_2 \int_0^{\frac{x}{\sqrt{4t}}} e^{-s^2} ds$$

Now use the initial condition, expressed as a limit as follows.

$$x > 0, \; 1 = \lim_{t \to 0+} U(\frac{x}{\sqrt{4t}}) = C_1 + C_2 \int_0^{+\infty} e^{-s^2} ds$$

$$x < 0, \; 0 = \lim_{t \to 0+} U(\frac{x}{\sqrt{4t}}) = C_1 + C_2 \int_0^{-\infty} e^{-s^2} ds$$

Note that $\int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}$, we have $C_1 = \frac{1}{2}, C_2 = \frac{1}{\sqrt{\pi}}$. Therefore,

$$u(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4t}}} e^{-s^2} ds$$

Method 2: The solution for the Cauchy problem is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$$

where $S(x,t)$ is the heat kernel, $S(x,t) = \frac{1}{\sqrt{4k\pi t}}e^{-\frac{x^2}{4kt}}$. Here, $k = 1$

$$\phi(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$$

Then,

$$u(x,t) = \int_0^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4t}}} e^{-s^2} ds.$$

6. (**10 points**) Derive the solution formula for the following initial-boundary value problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < +\infty, \quad t > 0 \\ u(x, t = 0) = \phi(x) & 0 < x < +\infty \\ \partial_x u(x = 0, t) = 0, & t > 0 \end{cases}$$

by the method of reflection.

Solution: Use the reflection method, and first consider the following Cauchy Problem:

$$\begin{cases} \partial_t v = \partial_x^2 v, & 0 < x < +\infty, \quad t > 0 \\ v(x, t = 0) = \phi_{even}(x) & 0 < x < +\infty \end{cases}$$

where $\phi_{even}(x)$ is even extension of ϕ which is given by

$$\phi_{even}(x) = \begin{cases} \phi(x), & \text{if } x > 0 \\ \phi(-x), & \text{if } x < 0 \end{cases}$$

Then the unique solution is given by:

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi_{even}(y)dy$$

And since $\phi_{even}(x)$ is even, so is $v(x,t)$ for $t > 0$,which implies

$$\partial_x v(x = 0, t) = 0, t > 0$$

Set $u(x,t) = v(x,t), x > 0$,then $u(x,t)$ is the unique solution of Neumann Problem on the half-line. More presicely, $x > 0, t > 0$

$$\begin{aligned} u(x,t) &= \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^{0} S(x-y,t)\phi(-y)dy \\ &= \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_0^{\infty} S(x+y,t)\phi(y)dy \\ &= \frac{1}{\sqrt{4k\pi t}} \int_0^{\infty} [e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}]\phi(y)dy. \end{aligned}$$